A Survey of Singularities

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“

That’s what I can tell you about singularities. They’re there.

”

Malcolm Perry

Outline

- The right way to define a singularity
- Penrose's 1965 singularity theorem
- Cosmic censorship, weak and strong

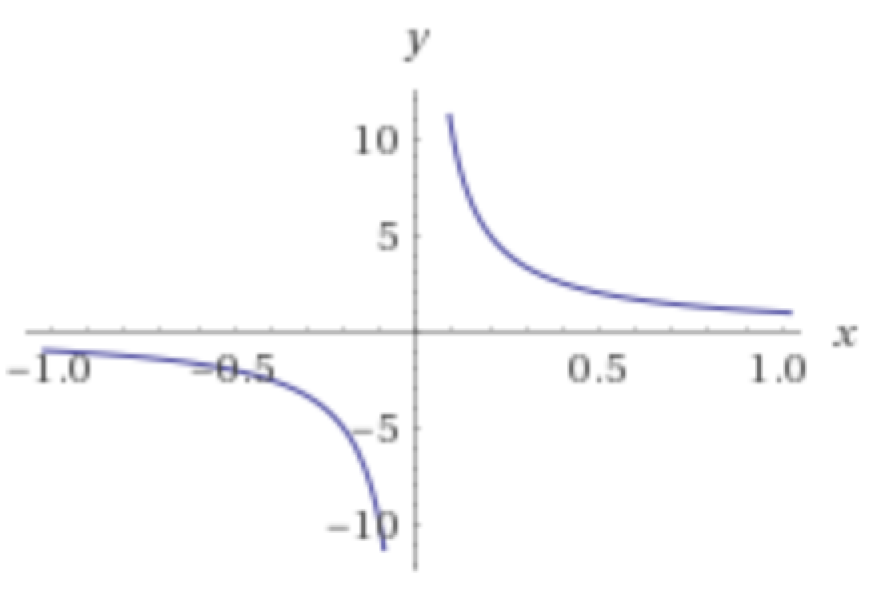
Intuition

- Singularities are places where things go wrong.
- Something becomes infinite or ill-defined.

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, t > 0$$

- Caution: under a simple change of coordinates, even Minkowski space can be made to look singular.

$$ds^2 = -\frac{1}{t'^2} dt'^2 + dx^2 + dy^2 + dz^2$$



Singularities, fake and real

- The Schwarzschild solution appears to have two singularities as r->0 and as r->2M.
- At r=0, a scalar invariant diverges. How to distinguish a divergence "in the spacetime" from a divergence "out at infinity"?

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

- Moreover, curvature singularity is too strong a requirement– could have singular spacetime with well-behaved scalar invariants.

Geodesic (in)completeness

- A spacetime is *geodesically complete* if any geodesic can be extended to arbitrarily large values of its affine parameter.
- Intuitively, no observer should be able to "go to infinity" in finite proper time.

$$g_{ab} = \Omega \eta_{ab}, \quad \Omega(t,x) = \begin{cases} 1 & |x| \geq 1 \\ (1 - 1/t^6)x^2 + 1/t^6 & |x| < 1. \end{cases}$$

$$-\frac{1}{t^6}\left(\frac{dt}{ds}\right)^2 = -1 \implies \frac{dt}{ds} = t^3 \implies \tau_0 = \int_0^{\tau_0} ds = \int_1^{\infty} \frac{dt}{t^3} = \frac{1}{2}.$$

- Timelike and null geodesic completeness are sufficient for a spacetime to be singularity-free, but still too strong.

Raychaudhuri's Singularity Theorem (1955)

- Suppose we have an (affinely parametrized) timelike geodesic vector field u.

$$u^\nu \nabla_\nu u^\mu = 0$$

- Raychaudhuri showed that along geodesics, the sign of the derivative of the divergence is determined by the curvature:

$$u^\nu \nabla_\nu (\nabla_\mu u^\mu) = -S - R_{\rho\sigma} u^\rho u^\sigma, S \geq 0.$$

- Suppose the *timelike convergence condition* holds:

$$R_{\rho\sigma} u^\rho u^\sigma \geq 0$$

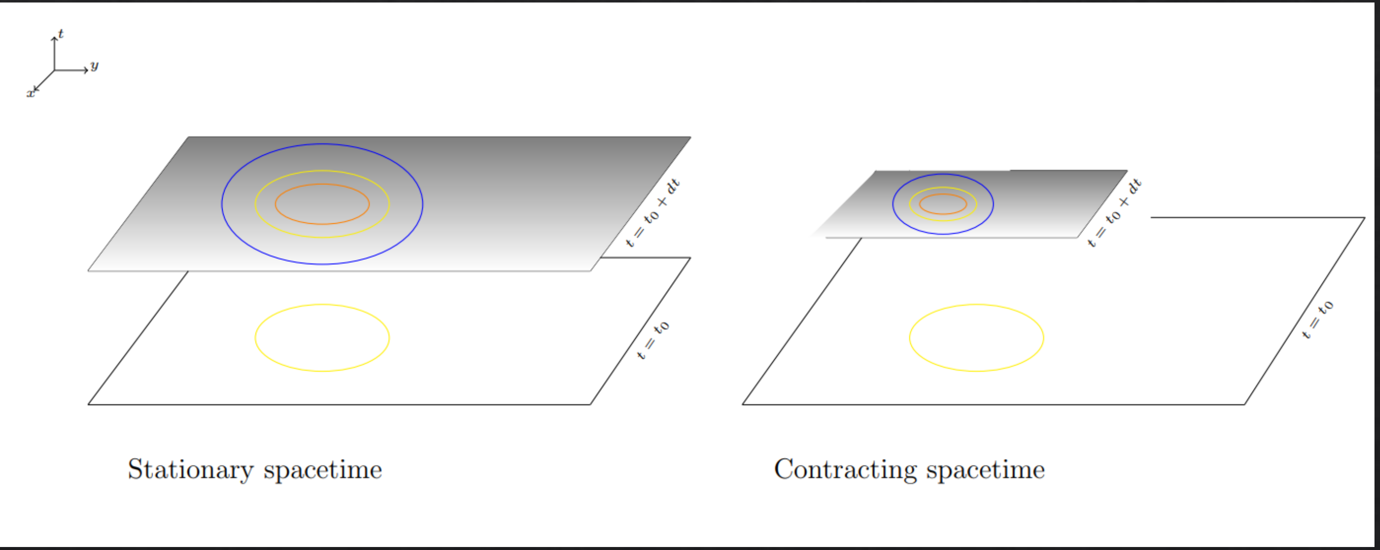
- Then the divergence of u blows up along geodesics, so all geodesics converge after a finite proper time.

Raychaudhuri and Komar

- Komar generalized Raychaudhuri's result for irrotational dust to fluids.
  - *Suppose $\Lambda=0$ and we have a perfect fluid with a geodesic, irrotational velocity vector field u. If the expansion is negative at an instant of time and the timelike convergence condition holds, then the energy density diverges in the finite future along every integral curve of u.*
- Key takeaways:
  - Rotation and acceleration can help us avoid singularities.
  - Raychaudhuri tells us exactly what diverges.

Penrose's Singularity Theorem (1965)

- One can show that the collapse of a spherically symmetric mass (e.g. dust) results in a singularity (cf. Oppenheimer-Snyder). But Penrose proved something stronger.
- Given:
  - Attractive gravity (strong energy condition with $\Lambda=0$)
  - An event horizon (closed future-trapped surface)
  - Boundary conditions on an "instant of time" (non-compact Cauchy hypersurface)
- Then there are future incomplete null geodesics, i.e. the spacetime has a singularity.



Stationary spacetime

Contracting spacetime

Hawking and Penrose

- Hawking argued that closed trapped surfaces are present in any expanding universe which is homogeneous and isotropic.
- Hawking and Ellis later showed that cosmic microwave background + singularity theorems = initial singularity in the past.
- Given:
  - Convergence condition on timelike/null vectors[1] (curvature condition)
  - No closed timelike curves (causality condition)
  - A closed trapped surface (initial/boundary conditions)
- Then the spacetime has incomplete causal (timelike/null) geodesics.

[1]And the so-called "generic condition"– tidal forces do not always point along geodesics

Cosmic censorship

- Two conjectures, both due to Penrose
- Weak cosmic censorship– singularities produced by gravitational collapse are necessarily hidden behind event horizons
  - Interesting connections to the weak gravity conjecture (cf. Crisford, Horowitz, and Santos 2017)
- Strong cosmic censorship– the maximal Cauchy development is the whole spacetime (generically inextendible).

Why we care

- In the end, Malcolm Perry was right. Most of our singularity theorems only predict the existence of singularities; they cannot tell us where they are or what precisely diverges.
- Singularities = boundaries of spacetime. Implications for causal structure of spacetime (cf. conformal diagrams)
- Divergences in electromagnetism led us to QED. Similarly, where cosmic censorship fails, maybe quantum gravity can save us. Singularities tell us where to look for quantum gravity.

Thank you for your time!

Sources

- Geroch, https://www.sciencedirect.com/science/article/pii/0003491668901449
- Senovilla and Garfinkle, https://arxiv.org/pdf/1410.5226.pdf
- Reall, https://physics.aps.org/articles/v11/6
- Hawking and Ellis, *The Large Scale Structure of Spacetime*

The Large Scale Structure of Singularity Theorems

- (Senovilla) *If a spacetime of sufficient differentiability satisfies*
  1. *a condition on the curvature*
  2. *a causality condition*
  3. *and appropriate initial and/or boundary conditions*

  then it contains endless but incomplete causal geodesics.